3.6 Compare and contrast different parametric and non-parametric approaches for estimating conditional volatility.

3.7 Calculate conditional volatility using parametric and non-parametric approaches.

- **Parametric approach for VaR**
  1. **Historical Standard Deviation**

   The most common example of the parametric method in estimating future volatility is based on calculating historical variance or standard deviation using "mean squared deviation". For example, the following equation is used to estimate future variance based on a window of the K most recent returns data.

   $$\sigma^2 = \frac{(r_{t-K}^2 + \cdots + r_{t-3}^2 + r_{t-2}^2 + r_{t-1}^2)}{K}$$

   **Decide the estimation window:**

   Use a shorter estimation window for the historical standard deviation method results in forecasts that are more volatile. This is in part due to the fact that each observation has more weight, and extreme observations therefore have a greater impact on the forecast. However, an advantage of using a smaller K for the estimation window is the model adapts more quickly to changes.

  2. **Exponential Smoothing – RiskMetrics™ and GARCH**

     1. **RiskMetrics™**

        The RiskMetrics™ and GARCH approaches are both exponential smoothing weighting methods. RiskMetrics™ is actually a special case of the GARCH approach. Both exponential smoothing methods are similar to the historical standard deviation approach because all three methods are parametric, attempt to estimate conditional volatility, use recent historical data, and apply a set of weights to past squared returns.

        The only major difference between the historical standard deviation approach and the two exponential smoothing approaches is with respect to the weights placed on historical returns that are used to estimate future volatility.

        The historical standard deviation approach assumes all K returns in the window are equally weighted. Conversely, the exponential smoothing methods place a higher weight on more recent data, and the weights decline exponentially to zero as returns become older.
The rate at which the weights change, or smoothness, is determined by a parameter $\lambda$ (known as the decay factor) raised to a power. The parameter $\lambda$ must fall between 0 and 1 (i.e., $0 < \lambda < 1$); however, most models use parameter estimates between 0.9 and 1 (i.e., $0.9 < \lambda < 1$).

The following figure illustrates the weights of the historical volatility for the historical standard deviation approach and RiskMetrics™ approach.

Using the RiskMetrics™ approach, conditional variance is estimated using the following formula:

$$\sigma_t^2 = (1 - \lambda)(\lambda^0\sigma_{t-1,\lambda}^2 + \lambda^1\sigma_{t-2,\lambda}^2 + \lambda^2\sigma_{t-3,\lambda}^2 + \ldots + \lambda^k\sigma_{t-K-1,\lambda}^2)$$

**Example 1: Calculate and compare RiskMetrics™ and Historical Standard Deviation**

Compare the weights of the volatility parameter using $\lambda = 0.97$, $\lambda = 0.92$, and $K = 75$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$(1 - \lambda)\lambda^t$</th>
<th>$(1 - \lambda)\lambda^t$</th>
<th>$1/K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0300</td>
<td>0.0800</td>
<td>0.0133</td>
</tr>
<tr>
<td>-1</td>
<td>0.0291</td>
<td>0.0736</td>
<td>0.0133</td>
</tr>
<tr>
<td>-2</td>
<td>0.0282</td>
<td>0.0677</td>
<td>0.0133</td>
</tr>
<tr>
<td>-3</td>
<td>0.0274</td>
<td>0.0623</td>
<td>0.0133</td>
</tr>
<tr>
<td>-4</td>
<td>0.0266</td>
<td>0.0573</td>
<td>0.0133</td>
</tr>
</tbody>
</table>

Using the RiskMetrics™ approach, calculate the weight for the most current historical return, $t=0$, when $\lambda = 0.97$.

- $t = 0, \lambda = 0.97 \Rightarrow (1 - \lambda)\lambda^t = (1 - 0.97) \times 0.97^0 = 0.0300$
- $t = -1, \lambda = 0.97 \Rightarrow (1 - \lambda)\lambda^t = (1 - 0.97) \times 0.97^1 = 0.0291$
- $t = -2, \lambda = 0.97 \Rightarrow (1 - \lambda)\lambda^t = (1 - 0.97) \times 0.97^2 = 0.0282$

Calculate the weight for the most recent return using historical standard deviation approach with $K=75$. All historical returns are equally weighted. Therefore, the weights will all be equal to $1/75 = 0.0133$. 

Answers:
2. GARCH

A more general exponential smoothing model is the GARCH model. This is a time-series model used by analysts to predict time-varying volatility. Volatility is measured with a general GARCH(p, q) model using the following formula:

\[ \sigma_t^2 = a + b_1 \sigma_{t-1}^2 + b_2 \sigma_{t-2}^2 + \cdots + b_p \sigma_{t-p}^2 + c_1 \sigma_{t-1} + c_2 \sigma_{t-2} + \cdots + c_q \sigma_{t-q} \]

where \( a, b_1 \ldots b_p, \) and \( c_1 \ldots c_q = \) parameters estimated using historical data with \( p \) lagged terms on historical returns squared and \( q \) lagged terms on historical volatility.

A GARCH(1,1) model would look like this: \( \sigma_t^2 = a + b_1 \sigma_{t-1}^2 + c_1 \sigma_{t-1} \)

- **Nonparametric approach for VaR**

  (1) **Historical Simulation Method**
  Under the historical simulation, all returns are weighted equally based on the number of observations in the estimation window \( 1/K \).

  (2) **Hybrid Approach**
  The hybrid approach uses historical simulation to estimate the percentiles of the return and weights that decline exponentially (similar to GARCH or RiskMetrics™). The following three steps are required to implement the hybrid approach.

  **Step 1. Assign weights** for historical realized returns to the most recent \( K \) returns using an exponential smoothing processes as follows:

  \[ \frac{1 - \lambda}{1 - \lambda^K}, \frac{1 - \lambda}{1 - \lambda}, \frac{1 - \lambda}{1 - \lambda^2}, \frac{1 - \lambda}{1 - \lambda^3}, \ldots, \frac{1 - \lambda}{1 - \lambda^{K-1}} \]

  **Step 2. Order the returns.**

  **Step 3. Determine the VaR** for the portfolio by starting with the lowest return and accumulating the weights until \( x \) percentage is reached. Linear interpolation may be necessary to achieve an exact \( x \) percentage.
**Example 2: Nonparametric Approach for VaR - Hybrid Approach (1)**

Suppose an analyst is using a hybrid approach to determine a 5% VaR with the most recent 100 observations (K=100) and $\lambda=0.96$.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Six Lowest Returns</th>
<th>Number of Past Periods</th>
<th>Hybrid Weight</th>
<th>Hybrid Cumulative Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-4.70%</td>
<td>2</td>
<td>0.0391</td>
<td>0.0391</td>
</tr>
<tr>
<td>2</td>
<td>-4.10%</td>
<td>5</td>
<td>0.0346</td>
<td>0.0736</td>
</tr>
<tr>
<td>3</td>
<td>-3.70%</td>
<td>55</td>
<td>0.0045</td>
<td>0.0781</td>
</tr>
<tr>
<td>4</td>
<td>-3.60%</td>
<td>25</td>
<td>0.0153</td>
<td>0.0934</td>
</tr>
<tr>
<td>5</td>
<td>-3.40%</td>
<td>14</td>
<td>0.0239</td>
<td>0.1173</td>
</tr>
<tr>
<td>6</td>
<td>-3.20%</td>
<td>7</td>
<td>0.0318</td>
<td>0.1492</td>
</tr>
</tbody>
</table>

Note that the data above are already ranked as described in Step 2 of the hybrid approach. Therefore, the six lowest returns out of the most recent 100 observations are listed in the above table.

1. Calculate the hybrid weight assigned to the lowest return, -4.70%
2. Calculate the hybrid weight assigned to the second lowest return, -4.10%
3. Calculate the initial VaR at the 5th percentile using the hybrid approach.

**Ans:**

\[ \frac{1 - \frac{\lambda}{1-\lambda}}{1 - \frac{\lambda}{1-\lambda}} \times 0.96^{2-1} = 0.0391 \]

\[ \frac{1 - \frac{\lambda}{1-\lambda}}{1 - \frac{\lambda}{1-\lambda}} \times 0.96^{5-1} = 0.0346 \]

(3) The lowest and second lowest returns have cumulative weights of 3.91% and 7.36%, respectively. Therefore, we must interpolate to obtain the 5% VaR percentile. The 5% VaR using the hybrid approach is calculated as:

\[-\text{VaR} \frac{-(-4.70) - 0.0391}{0.0736 - 0.0391} \Rightarrow \text{VaR} = 4.51\% \]

**Example 3: Nonparametric Approach for VaR - Hybrid Approach (2)**

Assume that over the next 20 days there are no extreme losses. Therefore, the six lowest returns will be the same returns in 20 days (detail data in the following table). Notice that the weights are less for these observations because they are now further away.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Six Lowest Returns</th>
<th>Number of Past Periods</th>
<th>Hybrid Weight</th>
<th>Hybrid Cumulative Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-4.70%</td>
<td>22</td>
<td>0.0173</td>
<td>0.0173</td>
</tr>
<tr>
<td>2</td>
<td>-4.10%</td>
<td>25</td>
<td>0.0153</td>
<td>0.0325</td>
</tr>
<tr>
<td>3</td>
<td>-3.70%</td>
<td>75</td>
<td>0.0020</td>
<td>0.0345</td>
</tr>
<tr>
<td>4</td>
<td>-3.60%</td>
<td>45</td>
<td>0.0068</td>
<td>0.0413</td>
</tr>
<tr>
<td>5</td>
<td>-3.40%</td>
<td>34</td>
<td>0.0106</td>
<td>0.0519</td>
</tr>
<tr>
<td>6</td>
<td>-3.20%</td>
<td>27</td>
<td>0.0141</td>
<td>0.0659</td>
</tr>
</tbody>
</table>

Calculate the revised VaR at the 5th percentile.

Ans: \[-\frac{-\text{VaR} \times (-3.60) - 0.0413}{0.0519 - 0.0413} \Rightarrow \text{VaR} = 0.44\% \]
(3) Multivariate Density Estimation (MDE)

Under the MDE model, conditional volatility for each market state or regime is calculated as follows:

$$\sigma_t^2 = \sum_{i=1}^{k} \omega(x_{t-i}) r_{t-i}^2$$

where

- $x_{t-i}$ = the vector of relevant variables describing the market state or regime at time $t-i$
- $\omega(x_{t-i})$ = the weight used on observation $t-i$, as a function of the “distance” if the state $x_{t-i}$ from the current state $x_t$

The kernel function, $\omega(x_{t-i})$, is used to measure the relative weight in terms of “near” or “distant” from the current state. The MDE model is very flexible in identifying dependence on state variables. Some examples of relevant state variable in an MDE model are the interest rate volatility dependent on the level of the interest rates or the term structure of the interest rates, equity volatility dependent on implied volatility, and exchange rate volatility dependent on the interest rate spreads.

Nonparametric vs. Parametric VaR Methods

Three common types of nonparametric methods used to estimate VaR are

1. Historical simulation (HS)
2. Multivariate Density Estimation (MDE)
3. Hybrid

**Advantages**

- Nonparametric methods do not require assumptions regarding the entire distributions of returns to estimate VaR.
- Fat tails, skewness, and other deviation from some assumed distribution are no longer a concern in the estimation process for nonparametric methods.
- MDE allows for weights to vary based on how relevant the data is to the current market environment, regardless of the timing of the most relevant data.
- MDE is very flexible in introducing dependence on economic variables.
- Hybrid approach does not require distribution assumptions because it uses a historical simulation approach with an exponential weighting scheme.

**Disadvantages**

- Data is used more efficiently with parametric methods than nonparametric methods. Thus, large sample sizes are required to precisely estimate volatility using HS.
- Separating the full sample of data into different market regimes reduces the amount of usable data for HS.
- MDE may lead to data snooping or over-fitting in identifying required assumptions regarding the weighting scheme identification of relevant conditioning variables, and the number of observations used to estimate volatility.
- MDE requires a large amount data that is directly related to the number of conditioning variables used on the model.